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On the existence of weak solutions to a coupled system
of two turbulent flows

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Abstract In this paper, we study a model problem for the stationary turbulent motion of two fluids in disjoint bounded domains Ω_1 and Ω_2 such that $\Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \emptyset$. The specific difficulty of this problem arises from the boundary condition which characterizes the interaction of the fluid motions along Γ .

We prove the existence of a weak solution to the problem under consideration which is more regular than the solution obtained in [3]. Moreover, we establish some regularity results for any weak solution. Our discussion is heavily based on the results in appendices 1 and 2 which seem to be of independent interest.

1. Introduction

Let Ω_1 and Ω_2 be bounded domains in \mathbb{R}^d ($d = 2$ or $d = 3$) such that

$$\begin{aligned} \Omega_1 \cap \Omega_2 &= \emptyset, & \Gamma &:= \bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \emptyset, \\ \partial\Omega_i &\text{ Lipschitz,} & \Gamma &\subset \partial\Omega_i \text{ relatively open } (i = 1, 2). \end{aligned}$$

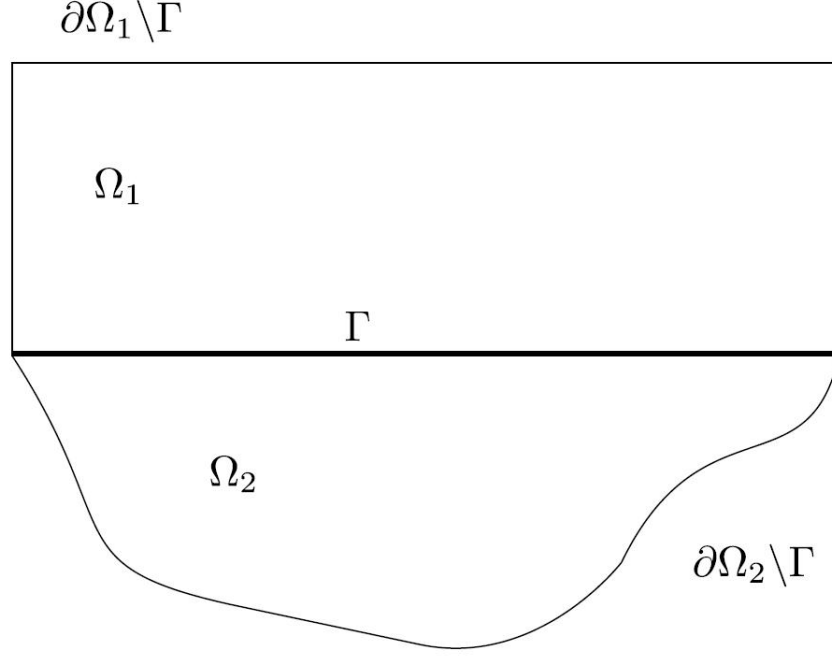


Figure 1

We consider the following system of PDEs in Ω_i ($i = 1, 2$)

$$(1.1) \quad -\operatorname{div}(\nu_i(k_i)\mathbf{D}(\mathbf{u}_i)) + \nabla p_i = \mathbf{f}_i \quad \text{in } \Omega_i,$$

$$(1.2) \quad \operatorname{div} \mathbf{u}_i = 0 \quad \text{in } \Omega_i,$$

$$(1.3) \quad -\Delta k_i = \mu_i(k_i)|\mathbf{D}(\mathbf{u}_i)|^2 \quad \text{in } \Omega_i$$

where

$\mathbf{u}_i = (u_{i1}, \dots, u_{id}) = \text{mean velocity}$, $p_i = \text{mean pressure}$,
 $k_i = \text{mean turbulent kinetic energy}$

are the unknown functions. For a vector field $\mathbf{u} = (u_1, \dots, u_d)$ we use the notations

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top), \quad |\mathbf{D}(\mathbf{u})|^2 = \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}).$$

The coefficients ν_i and μ_i are assumed to be uniformly bounded. We notice that the special case $\nu_i(k_i) = \nu_{i0} + \nu_{iT}(k_i)$ where

$$\nu_{i0} = \text{const} > 0 \quad \text{dynamic viscosity of the fluid,}$$

$$0 \leq \nu_{iT}(k_i) \leq \text{const} \quad \text{eddy viscosity,}$$

as well as the two cases

$$\mu_i(k_i) = \nu_i(k_i) \quad \text{or} \quad \mu_i(k_i) = \nu_{iT}(k_i) \quad ^1)$$

are included in our discussion.

Finally, \mathbf{f}_i represents an external force in Ω_i .

The system (1.1) - (1.3) belongs to the class of one-equation RANS (Reynolds Averaged Navier-Stokes) models. The triple (\mathbf{u}_i, k_i, p_i) ($i = 1, 2$) characterizes the stationary turbulent motion of a viscous fluid in Ω_i , where the convection term in the fluid equations as well as in the turbulent kinetic energy equations is neglected.

A discussion of RANS models can be found in [2; pp. 304-316], [12; pp. 182-196, 216-252], [18; 319-337] (with $\mu(k) = \nu_T(k)$), and in [14] within the context of oceanography. Related problems (but without turbulence effects) are studied in [17]. The stationary turbulent motion of a fluid with unbounded eddy viscosities of the type $\nu_T(k) = c_0 \sqrt{k}$ (Kolmogorov 1942, Prandtl 1945) has been studied in [7] and [13]. ■

We complete (1.1) - (1.3) by the following boundary conditions which link both systems of PDEs in Ω_1 and Ω_2 through the interface Γ :

$$(1.4) \quad \begin{cases} \mathbf{u}_i = \mathbf{0} & \text{on } \partial\Omega_i \setminus \Gamma, \\ \mathbf{u}_i \cdot \mathbf{n}_i = 0 & \text{on } \Gamma, \\ \nu_i(k_i)(\mathbf{D}(\mathbf{u}_i)\mathbf{n}_i)_\tau + |\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j)_\tau = 0 & \text{on } \Gamma \ (i \neq j), \end{cases}$$

¹⁾If $\mu_i = \nu_i$, system (1.1), (1.3) has some common features with the thermistor equations (see, e. g., Howison, S. D.; Rodrigues, J. F.; Shillor, M., *Stationary solutions to the thermistor problem*. J. Math. Analysis Appl. 174 (1993), 573-588; Cimatti, G., *The stationary thermistor problem with a current limiting device*. Proc. Royal Soc. Edinb. 116A (1990), 79-84). We notice that the assumption $\mu_i = \nu_i$ significantly simplifies the arguments of the passage to the limit in (1.3) with approximate solutions (cf. [7] and Gallouët, T.; Lederer, J.; Lewandowski, R.; Murat, F.; Tartar, L., *On a turbulent system with unbounded eddy viscosities*. Nonlin. Analysis 52 (2003), 1051-1068).

$$(1.5) \quad k_i = 0 \quad \text{on} \quad \partial\Omega_i \setminus \Gamma, \quad k_i = G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) \quad \text{on} \quad \Gamma$$

where

$$\begin{aligned} \mathbf{n}_i &= (n_{i1}, \dots, n_{id}) = \text{unit outward normal on } \partial\Omega_i, \\ \boldsymbol{\xi}_\tau &= \boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \mathbf{n}_i) \mathbf{n}_i \quad (\boldsymbol{\xi} \in \mathbb{R}^d), \\ (1.6) \quad 0 &\leq G_i(t) \leq c_0 t, \quad |G_i(t) - G_i(\bar{t})| \leq c_0 |t - \bar{t}| \quad \forall t, \bar{t} \in [0, +\infty) \quad (c_0 = \text{const} > 0) \end{aligned}$$

($i = 1, 2$). In (1.4), the boundary conditions on the (fixed) interface Γ model the situation when the interface is nonpermeable for both fluids which, however, do not completely adhere to the interface. Along this interface the fluids exhibit a partial slip which produces kinetic energy (cf. [3; pp. 69-73] for more details).

The boundary value problem (1.1) - (1.5) (with $\nabla \mathbf{u}_i$ in place of $\mathbf{D}(\mathbf{u}_i)$ in (1.1), (1.3) and (1.4)) has been investigated in [3]. In this paper, the authors prove the existence of a solution $\{\mathbf{u}_1, k_1, p_1; \mathbf{u}_2, k_2, p_2\}$ to (1.1)-(1.5) where (1.1) is satisfied in the usual weak sense (cf. our definition in Section 2), while (1.4) is satisfied in the sense of transposition of the Laplacean $-\Delta$ under zero boundary conditions. The aim of the present paper is to give an existence proof for a weak solution to (1.1)-(1.5) (in the sense of the definition of Section 2). Our proof is shorter and more transparent than the one in [3]. Moreover, we establish some regularity results on (\mathbf{u}_i, k_i) .

Our paper is organized as follows. In Section 2, we introduce the notion of weak solution $\{\mathbf{u}_1, k_1; \mathbf{u}_2, k_2\}$ to (1.1)-(1.5). By appealing to standard references, we show the existence of a pressure p_i associated with the pair (\mathbf{u}_i, k_i) ($i = 1, 2$). Section 3 contains our main existence result. It's proof is based on a straightforward application of the Schauder ²⁾ fixed point theorem. A higher integrability result on $\nabla \mathbf{u}_i$ is established in Section 4. From this result we deduce the local existence of the second order derivatives of k_i . In Appendix 1 we study in great detail the problem of whether a function which belongs to a Sobolev-Slobodeckij space over Γ and equals zero on $\partial\Omega \setminus \Gamma$, is a trace of a Sobolev function defined in Ω . The solution of this problem is fundamental to the homogenization of the boundary condition (1.5). Finally, Appendix 2 is concerned with the inhomogeneous Dirichlet problem for the Poisson equation with right hand side in L^1 .

2. Weak formulation of (1.1)-(1.5)

Let $W^{1,q}(\Omega)$ ($1 \leq q < +\infty$) denote the usual Sobolev space. We define

$$W_0^{1,q}(\Omega) := \{\varphi \in W^{1,q}(\Omega) : \varphi = 0 \quad \text{a. e. on} \quad \partial\Omega\}.$$

²⁾We notice that the Schauder fixed point theorem has been also used in: Bernardi, C.; Chacon, T.; Lewandowski, R.; Murat, F., *Existence d'une solution pour un modèle de deux fluides turbulents couplés*. C. R. Acad. Sci. Paris, Ser. I, 328 (1999), 993-998. In comparison with this paper, our existence theorem for a weak solution $\{\mathbf{u}_1, k_1, p_1; \mathbf{u}_2, k_2, p_2\}$ to (1.1)-(1.5) (see Section 3) involves more regularity of k_1, k_2 (see Remark 2.2 for details).

Spaces of vector-valued function will be denoted by bold letters, e. g., $\mathbf{L}^q(\Omega) := [L^q(\Omega)]^d$, $\mathbf{W}^{1,q}(\Omega) := [W^{1,q}(\Omega)]^d$ etc. Next, define

$$\begin{aligned} \mathbf{V}_i &:= \{ \mathbf{v} \in \mathbf{W}^{1,2}(\Omega_i) : \operatorname{div} \mathbf{v} = 0 \quad \text{a. e. in } \Omega_i, \\ &\quad \mathbf{v} = \mathbf{0} \quad \text{a. e. on } \partial\Omega_i \setminus \Gamma, \quad \mathbf{v} \cdot \mathbf{n}_i = 0 \quad \text{a. e. on } \Gamma \} \end{aligned}$$

($i = 1, 2$).

Without any further reference, throughout the paper we suppose

$$\left[\begin{array}{l} \text{there exist constants } \nu_*, \nu^* \text{ and } \mu^* \text{ such that} \\ 0 < \nu_* \leq \nu_i(t) \leq \nu^* < +\infty, \quad 0 \leq \mu_i(t) \leq \mu^* < +\infty \quad \forall t \in \mathbb{R} \end{array} \right] \quad (i = 1, 2).$$

Definition Let $\mathbf{f}_i \in \mathbf{L}^{2^*}(\Omega_i)$ ³⁾ ($i = 1, 2$). The functions $\{\mathbf{u}_1, k_1; \mathbf{u}_2, k_2\}$ are called weak solution to (1.1)-(1.5) if

$$(2.1) \quad (\mathbf{u}_i, k_i) \in \mathbf{V}_i \times \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1,q}(\Omega_i) \quad (i = 1, 2),$$

$$(2.2) \quad \left\{ \begin{array}{l} \int_{\Omega_1} \nu_1(k_1) \mathbf{D}(\mathbf{u}_1) : \mathbf{D}(\mathbf{v}_1) + \int_{\Omega_2} \nu_2(k_2) \mathbf{D}(\mathbf{u}_2) : \mathbf{D}(\mathbf{v}_2) + \\ + \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2| (\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) dS = \\ = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1 + \int_{\Omega_2} \mathbf{f}_2 \cdot \mathbf{v}_2 \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2, \end{array} \right.$$

$$(2.3) \quad \left\{ \begin{array}{l} \text{for some } r > d, \\ \int_{\Omega_i} \nabla k_i \cdot \nabla \varphi = \int_{\Omega_i} \mu_i(k_i) |\mathbf{D}(\mathbf{u}_i)|^2 \varphi \quad \forall \varphi \in W_0^{1,r}(\Omega_i) \end{array} \right. \quad \text{⁴⁾},$$

$$(2.4) \quad k_i = 0 \quad \text{a. e. on } \partial\Omega_i \setminus \Gamma, \quad k_i = G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) \quad \text{a. e. on } \Gamma.$$

Remark 2.1 (existence of a pressure) Define

³⁾By q^* we denote Sobolev embedding exponent for $W^{1,q}(\Omega)$ ($\Omega \subset \mathbb{R}^N$ bounded, Lipschitzian; $N \geq 2$), i. e. $q^* = \frac{Nq}{N-q}$ if $1 \leq q < N$, and $1 \leq q^* < +\infty$ if $q = N$. If $q > N$, then $W^{1,q}(\Omega) \subset C(\bar{\Omega})$ continuously.

⁴⁾Notice that $r > N$ iff $1 < r < \frac{N}{N-1}$.

$$\mathbf{W}_{0,\Gamma}^{1,2}(\Omega_i) := \{\mathbf{w} \in \mathbf{W}^{1,2}(\Omega_i) : \begin{aligned} \mathbf{w} &= \mathbf{0} & \text{a. e. on } \partial\Omega_i \setminus \Gamma, \\ \mathbf{w} \cdot \mathbf{n}_i &= 0 & \text{a. e. on } \Gamma \end{aligned}\}$$

($i = 1, 2$). Clearly, \mathbf{V}_i is a closed subspace of $\mathbf{W}_{0,\Gamma}^{1,2}(\Omega_i)$. We have:

Let $\{\mathbf{u}_1, k_1; \mathbf{u}_2, k_2\}$ be a weak solution to (1.1)-(1.5). Then there exists $p_i \in L^2(\Omega_i)$ with $\int_{\Omega_i} p_i = 0$ such that

$$(2.2') \quad \begin{cases} \int_{\Omega_i} \nu_i(k_i) \mathbf{D}(\mathbf{u}_i) : \mathbf{D}(\mathbf{w}) + (-1)^{i+1} \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2| (\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{w} dS = \\ = \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{w} + \int_{\Omega_i} p_i \operatorname{div} \mathbf{w} \quad \forall \mathbf{w} \in \mathbf{W}_{0,\Gamma}^{1,2}(\Omega_i). \end{cases}$$

In addition, there holds

$$(2.2'') \quad \|p_i\|_{L^2} \leq c \left(\|\nabla \mathbf{u}_i\|_{L^2} + \|\mathbf{f}_i\|_{L^{2^*}} \right).$$

To prove this, we first note the following

Proposition Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded Lipschitz domain and let $1 < r < +\infty$. Then, for every $f \in L^r(\Omega)$ with $\int_{\Omega} f = 0$, there exists $\mathbf{v} \in \mathbf{W}_0^{1,r}(\Omega)$ such that

$$\operatorname{div} \mathbf{v} = f \quad \text{a. e. in } \Omega,$$

$$\|\nabla \mathbf{v}\|_{L^r} \leq c \|f\|_{L^r}.$$

For a proof, see, e. g. [9; Chap. III, Thm. 3.2], [22; Chap. II, Lemma 2.1.1, a)].

We now proceed as follows. For $\mathbf{w} \in \mathbf{W}_{0,\Gamma}^{1,2}(\Omega_i)$, define

$$\mathcal{F}_i(\mathbf{w}) := \int_{\Omega_i} \nu_i(k_i) \mathbf{D}(\mathbf{u}_i) : \mathbf{D}(\mathbf{w}) + (-1)^{i+1} \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2| (\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{w} dS - \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{w}$$

($i = 1, 2$). It is easy to check that \mathcal{F}_i is a linear continuous functional on $\mathbf{W}_{0,\Gamma}^{1,2}(\Omega_i)$. By (2.2), $\mathcal{F}_i(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{V}_i$.

Next, the above Proposition implies that the mapping

$$A : \mathbf{v} \mapsto A\mathbf{v} = \operatorname{div} \mathbf{v}$$

is surjective from $\mathbf{W}_{0,\Gamma}^{1,2}(\Omega_i)$ onto the space

$$\left\{ f \in L^2(\Omega_i) : \int_{\Omega_i} f = 0 \right\}.$$

Now, following word by word the arguments of the proof in [9; Chap. III, Thm. 5.2] or [22; Chap. II, Lemma 2.11, b)] we obtain the existence of a $p_i \in L^2(\Omega_i)$ with $\int_{\Omega_i} p_i = 0$ such that

$$\mathcal{F}_i(\mathbf{w}) = \int_{\Omega_i} p_i \operatorname{div} \mathbf{w} \quad \forall \mathbf{w} \in \mathbf{W}_{0,\Gamma}^{1,2}(\Omega_i),$$

i. e., (2.2') holds.

Estimate (2.2'') is readily seen.

Remark 2.2 In [3; Thm. 5.2, pp. 88-89] the notion of (weak) solution to (1.1)-(1.5) means that k_i belongs to the Sobolev-Slobodeckij space $W^{s,2}(\Omega_i)$ ($0 < s < \frac{1}{2}$), and that (1.3) is satisfied in the sense of transposition of $-\Delta$ (cf. [3; p. 78]). In contrast to that paper, our definition of weak solution to (1.1)-(1.5) involves more regularity of k_i ⁵⁾.

Indeed, for any $0 < s < \frac{1}{2}$ we have $\frac{2d}{2+d-2s} < \frac{d}{d-1}$. Thus, if

$$\frac{2d}{2d-2s} < q < \frac{d}{d-1},$$

then

$$1 - \frac{d}{q} > s - \frac{d}{2},$$

and therefore

$$W^{1,q}(\Omega_i) \subset W^{s,2}(\Omega_i)$$

(see, e. g., [24; p. 328]). Hence, $k_i \in \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1,q}(\Omega_i)$ implies $k_i \in W^{s,2}(\Omega_i)$ for all

$$0 < s < \frac{1}{2}.$$

⁵⁾See also Appendix 2.

Finally, let $k_i \in W^{1,q}(\Omega_i)$ ($1 \leq q < \frac{d}{d-1}$) satisfy (2.3) and (2.4). Integration by parts on the left hand side of (2.3) gives, for any $\varphi \in W^{2,2}(\Omega_i) \cap W_0^{1,2}(\Omega_i)$,

$$-\int_{\Omega_i} k_i \Delta \varphi + \int_{\Gamma} G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) \mathbf{n}_i \cdot \nabla \varphi dS = \int_{\Omega_i} \mu_i(k_i) |\mathbf{D}(\mathbf{u}_i)|^2 \varphi,$$

i. e., k_i satisfies (1.3) in the sense of transposition of $-\Delta$ under zero boundary conditions on φ (cf. [3; p. 78]). ■

3. Existence of a weak solution

The following theorem is the main result of our paper.

Theorem *Let $\Omega_i \subset \mathbb{R}^d$ ($i = 1, 2$; $d = 2$ or $d = 3$) be bounded domains of class \mathcal{C}^1 ⁶⁾. Suppose that assumption (A) ⁷⁾ is satisfied.*

Then, for every $\mathbf{f}_i \in \mathbf{L}^{2^}(\Omega_i)$ ($i = 1, 2$) there exists a weak solution $\{\mathbf{u}_1, k_1; \mathbf{u}_2, k_2\}$ to (1.1)-(1.5). In addition,*

$$(3.1) \quad k_i \geq 0 \quad \text{a. e. in } \Omega_i,$$

$$(3.2) \quad \sum_{i=1}^2 \|\mathbf{u}_i\|_{\mathbf{W}^{1,2}(\Omega_i)}^2 + \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2|^3 dS \leq c \sum_{j=1}^2 \|\mathbf{f}_j\|_{\mathbf{L}^{2^*}(\Omega_j)}^2,$$

$$(3.3) \quad \left\{ \begin{array}{l} \text{for every } 1 \leq q < \frac{d}{d-1} \quad \text{there exists } c = \text{const} \quad \text{such that} \\ \|\mathbf{k}_i\|_{\mathbf{W}^{1,q}(\Omega_i)} \leq c \sum_{j=1}^2 \|\mathbf{f}_j\|_{\mathbf{L}^{2^*}(\Omega_j)}^2 \\ \text{where } c = c(q) \rightarrow +\infty \quad \text{as } q \rightarrow \frac{d}{d-1}, \end{array} \right.$$

$$(3.4) \quad \left\{ \begin{array}{l} \text{for every } \Omega'_i \Subset \Omega_i \quad \text{and every } \delta > 0, \\ \int_{\Omega'_i} \frac{|\nabla k_i|^2}{(1+k_i)^{1+\delta}} \leq \frac{c}{\delta} \sum_{j=1}^2 \|\mathbf{f}_j\|_{\mathbf{L}^{2^*}(\Omega_j)}^2, \\ \text{where } c \rightarrow +\infty \quad \text{as } \text{dist}(\Omega'_i, \partial\Omega_i) \rightarrow 0. \end{array} \right.$$

⁶⁾The condition $\Omega_i \in \mathcal{C}^1$ we need in order to apply Theorem A2.1.

⁷⁾See p. 22

Proof We consider the space $L^1(\Omega_1) \times L^1(\Omega_2)$ equipped with the norm

$$\|(k_1, k_2)\| := \sum_{i=1}^2 \|k_i\|_{L^1(\Omega_i)}.$$

For appropriate $R > 0$ which will be fixed below, we set

$$\mathcal{K}_R := \{(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2) : \|(k_1, k_2)\| \leq R\}.$$

Then, for any $(k_1, k_2) \in \mathcal{K}_R$ we show that there exists exactly one $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$ which satisfies (2.2). With $(\mathbf{u}_1, k_1; \mathbf{u}_2, k_2)$ at hand, we deduce from Theorem A2.1 the existence and uniqueness of a pair $(\hat{k}_1, \hat{k}_2) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ ($1 < q < \frac{d}{d-1}$ arbitrary) which solves (2.3) with the given L^1 -function $\mu_i(k_i)|D(\mathbf{u}_i)|^2$ on the right hand side, and with given $G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2)$ on Γ ($i = 1, 2$). This gives rise to introduce a mapping $\mathcal{T} : \mathcal{K}_R \rightarrow \mathcal{K}_R$ by

$$\mathcal{T}(k_1, k_2) := (\hat{k}_1, \hat{k}_2).$$

We then prove:

- (i) \mathcal{T} is continuous;
- (ii) $\mathcal{T}(\mathcal{K}_R)$ is precompact.

From Schauder's fixed it follows that there exists $(k_1^*, k_2^*) \in \mathcal{K}_R$ such that $\mathcal{T}(k_1^*, k_2^*) = (k_1^*, k_2^*)$.

Now, with the fixed point $(\mathbf{k}_1^*, \mathbf{k}_2^*)$ at hand, we obtain the existence and uniqueness of a pair $(\mathbf{u}_1^*, \mathbf{u}_2^*) \in \mathbf{V}_1 \times \mathbf{V}_2$ which satisfies (2.2) (with $(\mathbf{k}_1^*, \mathbf{k}_2^*)$ in place of (k_1, k_2) therein). By the definition of \mathcal{T} , the functions $\{\mathbf{u}_1^*, k_1^*; \mathbf{u}_2^*, k_2^*\}$ are a weak solution to (1.1)-(1.5). ■

We turn to the details of the proof.

Definition of $\mathcal{T} : \mathcal{K}_R \rightarrow \mathcal{K}_R$. The space $\mathbf{V}_1 \times \mathbf{V}_2$ is a Hilbert space with respect to the scalar product

$$\langle (\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle := \sum_{i=1}^2 \int_{\Omega} \nabla \mathbf{u}_i \cdot \nabla \mathbf{v}_i.$$

By $\|\cdot\| := \langle \cdot, \cdot \rangle^{\frac{1}{2}}$ we denote the associated norm.

1) *The mapping $(k_1, k_2) \mapsto (\mathbf{u}_1, \mathbf{u}_2)$.* Given any $(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$, we prove the existence and uniqueness of a pair $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$ which satisfies (2.2). To do this, we replace (2.2) by an operator equation in $\mathbf{V}_1 \times \mathbf{V}_2$ to which an abstract existence and uniqueness theorem applies.

Firstly, for any (fixed) $(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$ we introduce a linear bounded mapping $\mathcal{A}_{(k_1, k_2)} : \mathbf{V}_1 \times \mathbf{V}_2 \rightarrow \mathbf{V}_1 \times \mathbf{V}_2$ by

$$\langle \mathcal{A}_{(k_1, k_2)}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle := \sum_{i=1}^2 \int_{\Omega_i} \nu_i(k_i) \mathbf{D}(\mathbf{u}_i) : \mathbf{D}(\mathbf{v}_i).$$

By Korn's equality,

$$\langle \mathcal{A}_{(k_1, k_2)}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{u}_1, \mathbf{u}_2) \rangle \geq c_0 \|(\mathbf{u}_1, \mathbf{u}_2)\|^2 \quad (c_0 = \text{const} > 0)$$

for all $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$ (c_0 independent of (k_1, k_2)).

Secondly, observing the continuity of the trace mapping $\gamma : W^{1,2}(\Omega) \rightarrow L^4(\partial\Omega)$ ($d = 2$ and $d = 3$; see, e. g., [8], [11], [24; pp. 281-282, 329-330]) we obtain, for every $(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$ ⁸⁾,

$$\begin{aligned} & \left| \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2| (\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) dS \right| \leq \\ & \leq \left(\int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2|^{\frac{8}{3}} dS \right)^{\frac{3}{4}} \left(\int_{\Gamma} |\mathbf{v}_1 - \mathbf{v}_2|^4 dS \right)^{\frac{1}{4}} \\ & \leq c \left(\sum_{i=1}^2 \|\mathbf{u}_i\|_{L^{\frac{8}{3}}(\partial\Omega_i)}^2 \right) \sum_{j=1}^2 \|\mathbf{v}_j\|_{L^4(\partial\Omega_j)}^9 \\ & \leq c \|(\mathbf{u}_1, \mathbf{u}_2)\|^2 \|(\mathbf{v}_1, \mathbf{v}_2)\|. \end{aligned}$$

We now introduce a (nonlinear) mapping $\mathcal{B} : \mathbf{V}_1 \times \mathbf{V}_2 \rightarrow \mathbf{V}_1 \times \mathbf{V}_2$ by

$$\langle \mathcal{B}(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle := \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2| (\mathbf{u}_1 - \mathbf{u}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) dS.$$

By elementary calculus,

$$\begin{aligned} & \langle \mathcal{B}(\mathbf{u}_1, \mathbf{u}_2) - \mathcal{B}(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2), (\mathbf{u}_1, \mathbf{u}_2) - (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \rangle \geq \\ & \geq \int_{\Gamma} (|\mathbf{u}_1 - \mathbf{u}_2|^2 - |\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2|^2) (|\mathbf{u}_1 - \mathbf{u}_2| - |\bar{\mathbf{u}}_1 - \bar{\mathbf{u}}_2|) dS \geq 0 \end{aligned}$$

⁸⁾ For notational simplicity, in this section we use the same notation for a function in $W^{1,q}(\Omega)$ and its trace.

⁹⁾ Throughout the paper, we denote by c positive constants which may change their numerical value but do not depend on the functions under consideration.

and

$$\|\mathcal{B}(\mathbf{u}_1, \mathbf{u}_2) - \mathcal{B}(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2)\| \leq c(\|\mathbf{u}_1, \mathbf{u}_2\| + \|\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2\|) \sum_{i=1}^2 \|\mathbf{u}_i - \bar{\mathbf{u}}_i\|_{\mathbf{W}^{1,2}(\Omega_i)}$$

for all $(\mathbf{u}_1, \mathbf{u}_2), (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$

Thus,

$$\begin{cases} \mathcal{A}_{(k_1, k_2)} + \mathcal{B} & \text{is continuous on the whole of } \mathbf{V}_1 \times \mathbf{V}_2 \\ & \text{and maps bounded sets into bounded sets,} \end{cases}$$

$$\begin{cases} \langle (\mathcal{A}_{(k_1, k_2)} + \mathcal{B})(\mathbf{u}_1, \mathbf{u}_2) - (\mathcal{A}_{(k_1, k_2)} + \mathcal{B})(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2), (\mathbf{u}_1, \mathbf{u}_2) - (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \rangle \geq \\ \geq c_0 \|\mathbf{u}_1, \mathbf{u}_2 - (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2)\|^2 \quad \forall (\mathbf{u}_1, \mathbf{u}_2), (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \in \mathbf{V}_1 \times \mathbf{V}_2. \end{cases}$$

From [27; Thm. 26.A, p. 557] it follows that for every $\mathbf{f}_i \in \mathbf{L}^{2^*}(\Omega_i)$ ($i=1,2$) there exists exactly one $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$ such that

$$(3.5) \quad \langle (\mathcal{A}_{(k_1, k_2)} + \mathcal{B})(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle = \sum_{i=1}^2 \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2,$$

i. e., (2.2) holds with the given $(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$. In addition, we have

$$(3.6) \quad \sum_{i=1}^2 \|\mathbf{u}_i\|_{\mathbf{W}^{1,2}(\Omega_i)}^2 + \int_{\Gamma} |\mathbf{u}_1 - \mathbf{u}_2|^3 dS \leq c \sum_{j=1}^2 \|\mathbf{f}_j\|_{\mathbf{L}^{2^*}(\Omega_j)}^2,$$

where the constant c does not depend on (k_1, k_2) .

2) *The mapping $(\mathbf{u}_1, \mathbf{u}_2) \mapsto (\hat{k}_1, \hat{k}_2)$.* Let $1 < q < \frac{d}{d-1}$. Let $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$ denote the solution to (3.5) (uniquely determined by $(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$) which has been obtained by the preceding step 1).

Define

$$\tilde{h}_i := \begin{cases} G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma \end{cases}$$

(G_i as in (1.6); $i = 1, 2$). By Corollary A1.1,

$$(3.7) \quad \tilde{h}_i \in W^{1-\frac{1}{q}, q}(\partial\Omega_i), \quad \|\tilde{h}_i\|_{W^{1-\frac{1}{q}, q}(\partial\Omega_i)} \leq c \sum_{j=1}^2 \|\mathbf{u}_j\|_{\mathbf{W}^{1,2}(\Omega_j)}^2.$$

Now, from Theorem A2.1 and Theorem A2.2, 1° we obtain the existence and uniqueness of a pair $(\hat{k}_1, \hat{k}_2) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ such that

$$(3.8) \quad \hat{k}_i \geq 0 \quad \text{a. e. in } \Omega_i,$$

$$(3.9) \quad \int_{\Omega_i} \nabla \hat{k}_i \cdot \nabla \varphi_i = \int_{\Omega_i} \mu_i(k_i) |\mathbf{D}(\mathbf{u}_i)|^2 \varphi_i \quad \forall \varphi_i \in W^{1,q'}(\Omega_i),$$

$$(3.10) \quad \hat{k}_i = \tilde{h}_i \quad \text{a. e. on } \partial\Omega_i,$$

$$(3.11) \quad \|\hat{k}_i\|_{W^{1,q}(\Omega_i)} \leq c \left(\|\mathbf{D}(\mathbf{u}_i)\|^2_{L^1(\Omega_i)} + \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \right),$$

$$(3.12) \quad \left\{ \begin{array}{l} \text{for every } \Omega'_i \Subset \Omega_i \text{ and every } \delta > 0, \\ \frac{\nabla \hat{k}_i}{(1 + \hat{k}_i)^{\frac{1+\delta}{2}}} \in \mathbf{L}^2(\Omega'_i), \\ \int_{\Omega_i} \frac{\nabla \hat{k}_i}{(1 + \hat{k}_i)^{1+\delta}} \leq \frac{c}{\delta} \left(\|\mathbf{D}(\mathbf{u}_i)\|^2_{L^1(\Omega_i)} + \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \right), \\ \text{where } c \rightarrow +\infty \text{ as } \text{dist}(\Omega'_i, \partial\Omega_i) \rightarrow 0. \end{array} \right.$$

We notice that the constants c in (3.7), (3.11) and (3.12) do not depend on (k_1, k_2) . By combining (3.7) and (3.11) we find

$$(3.13) \quad \|(\hat{k}_1, \hat{k}_2)\| \leq c \sum_{i=1}^2 \|\mathbf{f}_i\|_{\mathbf{L}^{2^*}(\Omega_i)}^2 =: R.$$

3) Let us consider \mathcal{K}_R ¹⁰⁾ with R as in (3.13). For $(k_1, k_2) \in \mathcal{K}_R$, define

$$\mathcal{T} : (k_1, k_2) \mapsto (\mathbf{u}_1, \mathbf{u}_2) \mapsto \mathcal{T}(k_1, k_2) := (\hat{k}_1, \hat{k}_2),$$

where $(\mathbf{u}_1, \mathbf{u}_2)$ is as in step 1), (\hat{k}_1, \hat{k}_2) as in step 2). Then \mathcal{T} is a well-defined (single valued) mapping of \mathcal{K}_R into itself.¹¹⁾

(i) **\mathcal{T} is continuous.** Let be $(k_{1m}, k_{2m}) \in \mathcal{K}_R$ ($m \in \mathbb{N}$) such that

$$k_{im} \rightarrow k_i \quad \text{strongly in } L^1(\Omega_i) \quad \text{as } m \rightarrow \infty \quad (i = 1, 2).$$

¹⁰⁾ Recall $\mathcal{K}_R := \{(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2) : \|(k_1, k_2)\| \leq R\}$.

¹¹⁾ In fact, \mathcal{T} maps the whole of $L^1(\Omega_1) \times L^1(\Omega_2)$ into \mathcal{K}_R .

Clearly, $(k_1, k_2) \in \mathcal{K}_R$. Without loss of generality, we may assume that

$$(3.14) \quad k_{im} \rightarrow k_i \quad \text{a. e. in } \Omega_i \quad \text{as } m \rightarrow \infty \quad (i = 1, 2).$$

We prove that

$$\mathcal{T}(k_{1m}, k_{2m}) \rightarrow \mathcal{T}(k_1, k_2) \quad \text{strongly in } L^1(\Omega_1) \times L^1(\Omega_2) \quad \text{as } m \rightarrow \infty.$$

To begin with, we introduce the following notation. For (k_{1m}, k_{2m}) , let $(\mathbf{u}_{1m}, \mathbf{u}_{2m}) \in \mathbf{V}_1 \times \mathbf{V}_2$ denote the uniquely determined solution of

$$(3.5_m) \quad \langle (\mathcal{A}_{(k_{1m}, k_{2m})} + \mathcal{B})(\mathbf{u}_{1m}, \mathbf{u}_{2m}), (\mathbf{v}_1, \mathbf{v}_2) \rangle = \sum_{i=1}^2 \int_{\Omega_i} \mathbf{f}_i \cdot \mathbf{v}_i \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2.$$

Clearly,

$$(3.6_m) \quad \sum_{i=1}^2 \|\mathbf{u}_{im}\|_{\mathbf{W}^{1,2}(\Omega_i)}^2 + \int_{\Gamma} |\mathbf{u}_{1m} - \mathbf{u}_{2m}|^3 dS \leq c \sum_{i=1}^2 \|\mathbf{f}_i\|_{\mathbf{L}^{2^*}(\Omega_i)}^2.$$

Analogously, for the limit element (k_1, k_2) , let $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$ denote the uniquely determined solution to (3.5). This solution satisfies (3.6).

We claim

$$(3.15) \quad (\mathbf{u}_{1m}, \mathbf{u}_{2m}) \rightarrow (\mathbf{u}_1, \mathbf{u}_2) \quad \text{strongly in } \mathbf{W}^{1,2}(\Omega_1) \times \mathbf{W}^{1,2}(\Omega_2) \quad \text{as } m \rightarrow \infty.$$

To prove this, we first note that from (3.6_m) it follows that there exists a subsequence $\{(\mathbf{u}_{1m_s}, \mathbf{u}_{2m_s})\} (s \in \mathbb{N})$ such that

$$(\mathbf{u}_{1m_s}, \mathbf{u}_{2m_s}) \rightarrow (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \quad \text{weakly in } \mathbf{W}^{1,2}(\Omega_1) \times \mathbf{W}^{1,2}(\Omega_2) \quad \text{as } s \rightarrow \infty.$$

Using the compactness of the embedding $W^{1,2}(\Omega) \subset L^r(\partial\Omega)$ ($1 \leq r < 4$; $d = 2$ resp. $d = 3$), we obtain

$$\langle \mathcal{B}(\mathbf{u}_{1m_s}, \mathbf{u}_{2m_s}), (\mathbf{v}_1, \mathbf{v}_2) \rangle \rightarrow \langle \mathcal{B}(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2$$

as $m \rightarrow \infty$. With the help of (3.14) the passage to the limit $s \rightarrow \infty$ in (3.5_m) gives

$$\langle (\mathcal{A}_{(k_1, k_2)} + \mathcal{B})(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle = \sum_{i=1}^2 \int_{\Omega} \mathbf{f}_i \cdot \mathbf{v}_i \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{V}_1 \times \mathbf{V}_2.$$

Comparing this and (3.5) we find $\bar{\mathbf{u}}_i = \mathbf{u}_i$ ($i = 1, 2$). Therefore the whole sequence $\{(\mathbf{u}_{1m}, \mathbf{u}_{2m})\}$ converges weakly in $\mathbf{W}^{1,2}(\Omega_1) \times \mathbf{W}^{1,2}(\Omega_2)$ to $(\mathbf{u}_1, \mathbf{u}_2)$.

We now form the difference between (3.5_m) and (3.5), and use the test function $\mathbf{v}_i = \mathbf{u}_{im} - \mathbf{u}_i$ ($i = 1, 2$). Observing the monotonicity of \mathcal{B} , we find

$$\begin{aligned} \nu_* \sum_{i=1}^2 \int_{\Omega_i} |\mathbf{D}(\mathbf{u}_{im} - \mathbf{u}_i)|^2 &\leq \sum_{i=1}^2 \int_{\Omega_i} \nu_i(k_{im})(\mathbf{D}(\mathbf{u}_{im}) - \mathbf{D}(\mathbf{u}_i)) : \mathbf{D}(\mathbf{u}_{im} - \mathbf{u}_i) \\ &\leq \sum_{i=1}^2 \int_{\Omega_i} (-\nu_i(k_{im}) + \nu_i(k_i)) \mathbf{D}(\mathbf{u}_i) : \mathbf{D}(\mathbf{u}_{im} - \mathbf{u}_i) \\ &\rightarrow 0 \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Whence (3.15).

Next, set $(\hat{k}_{1m}, \hat{k}_{2m}) := \mathcal{T}(k_{1m}, k_{2m})$ ($m \in \mathbb{N}$) and $(\hat{k}_1, \hat{k}_2) := \mathcal{T}(k_1, k_2)$. Let $1 < q < \frac{d}{d-1}$. By the definition of \mathcal{T} , the pair $(\hat{k}_{1m}, \hat{k}_{2m}) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ is uniquely determined by (k_{1m}, k_{2m}) and $(\mathbf{u}_{1m}, \mathbf{u}_{2m})$ through

$$(3.9_m) \quad \int_{\Omega_i} \nabla \hat{k}_{im} \cdot \nabla \varphi_i = \int_{\Omega_i} \mu_i(k_{im}) |\mathbf{D}(\mathbf{u}_{im})|^2 \varphi_i \quad \forall \varphi_i \in W_0^{1,q'}(\Omega_i),$$

$$(3.10_m) \quad \hat{k}_{im} = \tilde{h}_{im} \quad \text{a. e. on } \partial\Omega_i,$$

where $\tilde{h}_{im} \in W^{1-\frac{1}{q},q}(\partial\Omega_i)$ is defined by

$$\tilde{h}_{im} := \begin{cases} G_i(|\mathbf{u}_{1m} - \mathbf{u}_{2m}|^2) & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma \end{cases}$$

(see Theorem A2.1). From (3.7) (with \mathbf{u}_{im} in place of \mathbf{u}_i) it follows that

$$\|\tilde{h}_{im}\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \leq c \sum_{j=1}^2 \|\mathbf{u}_{jm}\|_{\mathbf{W}^{1,2}(\Omega_j)}^2 \leq \text{const.}$$

We obtain

$$(3.16) \quad \tilde{h}_{im} \rightarrow \tilde{h}_i \quad \text{weakly in } W^{1-\frac{1}{q},q}(\partial\Omega_i) \quad \text{as } m \rightarrow \infty,$$

where \tilde{h}_i is defined as above, i. e.

$$\tilde{h}_i := \begin{cases} G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma \end{cases}$$

($i = 1, 2$). To see (3.16), we first note that (3.15) implies $\mathbf{u}_{im} \rightarrow \mathbf{u}_i$ strongly in $\mathbf{L}^4(\partial\Omega_i)$ as $m \rightarrow \infty$ ($d = 2$ resp. $d = 3$). Therefore

$$G_i(|\mathbf{u}_{1m} - \mathbf{u}_{2m}|^2) \rightarrow G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) \quad \text{strongly in } L^2(\Gamma) \quad \text{as } m \rightarrow \infty.$$

Since $W^{1-\frac{1}{q},q}(\partial\Omega_i)$ is reflexive, (3.16) is now readily seen by routine arguments.

To proceed, we note that \hat{k}_{im} satisfies the estimate

$$\begin{aligned}\|\hat{k}_{im}\|_{W^{1,q}(\Omega_i)} &\leq c\left(\| |\mathbf{D}(\mathbf{u}_{im})|^2 \|_{L^1(\Omega_i)} + \|\tilde{h}_{im}\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)}\right) \quad [\text{cf. (3.11)}] \\ &\leq c \sum_{j=1}^2 \|\mathbf{u}_{jm}\|_{\mathbf{W}^{1,2}(\Omega_j)}^2 \\ &\leq c \sum_{j=1}^2 \|\mathbf{f}_j\|_{\mathbf{L}^{2^*}(\Omega_j)}^2 \quad [\text{by (3.6}_m\text{)}]\end{aligned}$$

($i = 1, 2$; $m \in \mathbb{N}$). Hence there exists a subsequence $\{\hat{k}_{im_t}\}$ ($t \in \mathbb{N}$) such that

$$\hat{k}_{im_t} \rightarrow \bar{k}_i \quad \text{weakly in } W^{1,q}(\Omega_i) \quad \text{as } t \rightarrow \infty.$$

Using (3.14), (3.15) and (3.16) the passage to the limit $t \rightarrow \infty$ in (3.9 _{m_t}) and (3.10 _{m_t}) gives

$$\begin{aligned}\int_{\Omega_i} \nabla \bar{k}_i \cdot \nabla \varphi_i &= \int_{\Omega_i} \mu_i(k_i) |\mathbf{D}(\mathbf{u}_i)|^2 \varphi_i \quad \forall \varphi_i \in W_0^{1,q'}(\Omega_i), \\ \bar{k}_i &= \tilde{h}_i \quad \text{a. e. on } \partial\Omega_i.\end{aligned}$$

Combining this and (3.9), (3.10) we get

$$\begin{aligned}\int_{\Omega_i} \nabla (\bar{k}_i - \hat{k}_i) \cdot \nabla \varphi_i &= 0 \quad \forall \varphi_i \in W_0^{1,q'}(\Omega_i), \\ \bar{k}_i - \hat{k}_i &= 0 \quad \text{a. e. on } \partial\Omega_i.\end{aligned}$$

By theorem A2.1, $\bar{k}_i = \hat{k}_i$ a. e. in Ω_i ($i = 1, 2$). It follows that the whole sequence $\{\hat{k}_{im}\}$ converges weakly in $W^{1,q}(\Omega_i)$ to \hat{k}_i as $m \rightarrow \infty$. Therefore, by the compactness of the embedding $W^{1,q}(\Omega) \subset L^1(\Omega)$,

$$\hat{k}_{im} \rightarrow \hat{k}_i \quad \text{strongly in } L^1(\Omega_i) \quad \text{as } m \rightarrow \infty,$$

i. e., \mathcal{T} is continuous.

(ii) $\mathcal{T}(\mathcal{K}_R)$ is **precompact**. Let $(\hat{k}_{1m}, \hat{k}_{2m}) \in \mathcal{T}(\mathcal{K}_R)$ ($m \in \mathbb{N}$). Then $(\hat{k}_{1m}, \hat{k}_{2m}) = \mathcal{T}(k_{1m}, k_{2m})$, where $(k_{1m}, k_{2m}) \in \mathcal{K}_R$. As above, let $(\mathbf{u}_{1m}, \mathbf{u}_{2m}) \in \mathbf{V}_1 \times \mathbf{V}_2$ denote the uniquely determined solutions to (3.5 _{m}). The existence and uniqueness argument used at the end of the proof of the continuity of \mathcal{T} (cf. Theorem A2.1), implies that $(\hat{k}_{1m}, \hat{k}_{2m}) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ and $1 < q < \frac{d}{d-1}$) and (3.9 _{m}) and (3.10 _{m}) hold. It follows that

$$\|\hat{k}_{im}\|_{W^{1,q}(\Omega_i)} \leq c \sum_{j=1}^2 \|\mathbf{f}_j\|_{\mathbf{L}^{2^*}(\Omega_j)}^2 \quad (i = 1, 2; m \in \mathbb{N})$$

(cf. above). By the compactness of the embedding $W^{1,q}(\Omega) \subset L^1(\Omega)$, there exists a subsequence $\{\hat{k}_{im_s}\}$ ($s \in \mathbb{N}$) and an element $(l_1, l_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$ such that

$$\hat{k}_{im_s} \rightarrow l_i \quad \text{strongly in } L^1(\Omega_i) \quad \text{as } s \rightarrow \infty,$$

i. e. $\mathcal{T}(\mathcal{K}_R)$ is precompact.

By Schauder's fixed point theorem, there exists $(k_1^*, k_2^*) \in \mathcal{K}_R$ such that $\mathcal{T}(k_1^*, k_2^*) = (k_1^*, k_2^*)$. The proof of the theorem is complete.

4. Regularity properties of weak solutions

In this section, we establish regularity properties for any weak solution $\{\mathbf{u}_1, k_1; \mathbf{u}_2, k_2\}$ to (1.1)–(1.5) (see Sect. 2 for the definition).

Theorem 4.1 (Local regularity) *Let $\mathbf{f}_i \in \mathbf{L}^2(\Omega_i)$ ($i = 1, 2$). Then there exists $\sigma > 2$ such that for every weak solution $\{\mathbf{u}_1, k_1; \mathbf{u}_2, k_2\}$ to (1.1)–(1.5) there holds*

$$\nabla \mathbf{u}_i \in \mathbf{L}_{\text{loc}}^\sigma(\Omega_i), \quad k_i \in W_{\text{loc}}^{2, \frac{\sigma}{2}}(\Omega_i).$$

Indeed, the local higher integrability of $\nabla \mathbf{u}_i$ follows from [6; Prop. 4.1]. It follows $|\mathbf{D}(\mathbf{u}_i)|^2 \in \mathbf{L}_{\text{loc}}^{\frac{\sigma}{2}}(\Omega_i)$. Then $k_i \in W_{\text{loc}}^{2, \frac{\sigma}{2}}(\Omega_i)$ is a consequence of Theorem A 2.1, (A2.7).

Theorem 4.2 (global higher integrability of $\nabla \mathbf{u}_i$) *Assume that*

$$\bar{\Gamma} \cap (\partial\Omega_i \setminus \Gamma) \quad \text{is Lipschitz} \quad (i = 1, 2)^{12)}$$

Let $\mathbf{f}_i \in \mathbf{L}^2(\Omega_i)$. Then there exists $\rho > 2$ such that for every weak solution $\{\mathbf{u}_1, k_1; \mathbf{u}_2, k_2\}$ to (1.1)–(1.5) there holds

$$\nabla \mathbf{u}_i \in \mathbf{L}^\rho(\Omega_i).$$

This result is a special case of [26; Thm. 2.1]. ■

We notice that the higher integrability of the gradient has been used in [3] for the uniqueness of the weak solution to (1.1)–(1.5) in the case $d = 2$. It has been also used in [4].

Appendix 1. Extension of a function $g \in W^{s,q}(\Gamma)$ by zero onto $\partial\Omega \setminus \Gamma$

.

¹²⁾ See [26; (1.24a), (1.24b)] for details.

[1] Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with Lipschitz boundary $\partial\Omega$. For $0 < s < 1$ and $1 < q < +\infty$ we consider the Sobolev-Slobodeckij space

$$W^{s,q}(\partial\Omega) := \left\{ w \in L^q(\partial\Omega) : \int_{\partial\Omega} \int_{\partial\Omega} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y < +\infty \right\}$$

with the norm

$$\|w\|_{W^{s,q}(\partial\Omega)} := \left(\|w\|_{L^q(\partial\Omega)}^q + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y \right)^{\frac{1}{q}}$$

(see, e. g., [8], [19] for details).

Let $\Gamma \subset \partial\Omega$ be **relatively open**. We have

[1.1] Let $w \in W^{s,q}(\partial\Omega)$. If $w = 0$ a. e. on $\partial\Omega \setminus \Gamma$, then

$$\begin{aligned} & \int_{\partial\Omega} \int_{\partial\Omega} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y = \\ (A1.1) \quad & = \int_{\Gamma} \int_{\Gamma} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y + \\ & + \int_{\Gamma} |w(y)|^q \left(\int_{\partial\Omega \setminus \Gamma} \frac{1}{|x - y|^{N-1+sq}} dS_x \right) dS_y \\ & + \int_{\partial\Omega \setminus \Gamma} \left(\int_{\Gamma} \frac{|w(x)|^q}{|x - y|^{N-1+sq}} dS_x \right) dS_y \end{aligned}$$

This follows from the additivity of the integral.

We notice that the second and third integral on the right hand side of (A1.1) are equal. Indeed, we have

$$\begin{aligned}
& \int_{\Gamma} \left(\int_{\partial\Omega \setminus \Gamma} \frac{|w(y)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y = \\
& = \int_{\partial\Omega \setminus \Gamma} \left(\int_{\Gamma} \frac{|w(y)|^q}{|x-y|^{N-1+sq}} dS_y \right) dS_x \quad [\text{by Fubini-Tonelli}] \\
\text{(A1.2)} \quad & = \int_{\partial\Omega \setminus \Gamma} \left(\int_{\Gamma} \frac{|w(x)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y
\end{aligned}$$

[change of notation of the variables x and y].

1.2 Let $g \in L^q(\Gamma)$ ($1 < q < +\infty$), let $0 < s < 1$ and assume that

$$\text{(A1.3)} \quad \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x-y|^{N-1+sq}} dS_x dS_y < +\infty,$$

$$\text{(A1.4)} \quad \int_{\Gamma} |g(y)|^q \left(\int_{\partial\Omega \setminus \Gamma} \frac{1}{|x-y|^{N-1+sq}} dS_x \right) dS_y < +\infty.$$

Define

$$\tilde{g} := \begin{cases} g & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega \setminus \Gamma. \end{cases}$$

Then $\tilde{g} \in W^{s,q}(\partial\Omega)$.

Indeed, firstly $\tilde{g} \in L^q(\partial\Omega)$. Secondly, from (A1.3) and (A1.4) it follows

$$\begin{aligned}
+\infty &> \int_{\Gamma} \left(\int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x - y|^{N-1+sq}} dS_x + \int_{\partial\Omega \setminus \Gamma} \underbrace{\frac{|\tilde{g}(x) - g(y)|^q}{|x - y|^{N-1+sq}}}_{\tilde{g}(x)=0} dS_x \right) dS_y \\
&+ \int_{\partial\Omega \setminus \Gamma} \left(\int_{\Gamma} \underbrace{\frac{|g(x) - \tilde{g}(y)|^q}{|x - y|^{N-1+sq}}}_{\tilde{g}(y)=0} dS_x + \underbrace{0}_{g(\tilde{x})=\tilde{g}(y)=0} \right) dS_y \text{ [observe(A1.2) with } g \text{ in place of } w] \\
&= \int_{\partial\Omega} \left(\int_{\partial\Omega} \frac{|\tilde{g}(x) - \tilde{g}(y)|^q}{|x - y|^{N-1+sq}} dS_x \right) dS_y.
\end{aligned}$$

■

Remark A1.1 Under the above assumptions, for $y \in \Gamma$ define

$$\omega(y) = \omega_{s,q}(y) := \int_{\partial\Omega \setminus \Gamma} \frac{1}{|x - y|^{N-1+sq}} dS_x.$$

We have

- 1) ω is continuous on Γ ,
- 2) $\omega(y) \leq \frac{\text{mes}(\partial\Omega \setminus \Gamma)}{(\text{dist}(y, \partial\Omega \setminus \Gamma))^{N-1+sq}} < +\infty$,
- 3) let $x_0 \in \partial\Omega \setminus \Gamma$, $\text{dist}(x_0, \Gamma) = 0$; if there exists $a_0 > 0$, $\rho_0 > 0$ such that $\text{mes}((\partial\Omega \setminus \Gamma) \cap B_\rho(x_0)) \geq a_0 \rho^{N-1}$ for all $0 < \rho \leq \rho_0$ ¹³⁾ then

$$\lim_{y \in \Gamma, y \rightarrow x_0} \omega(y) = +\infty.$$

Condition (A1.4) reads

$$(A1.4') \quad \int_{\Gamma} \omega(y) |g(y)|^q dS_y < +\infty.$$

Thus, condition (A1.4) (resp. (A1.4')) expresses a decay property of g near the boundary $\partial\Gamma$. ■

¹³⁾ $B_\rho(x_0) = \{\xi \in \mathbb{R}^N : |\xi - x_0| < \rho\}$ We notice that the condition on $\text{mes}((\partial\Omega \setminus \Gamma) \cap B_\rho(x_0))$ occurs in the discussion of Campanato spaces; (see [8; pp. 209-245], [10; p. 32]) for more details.

The above discussion gives rise to introduce the following

Definition *Let $0 < s < 1$, let $1 < q < +\infty$ and let be ω as in Remark A1.1. Then*

$$W_{00}^{s,q}(\Gamma) := \left\{ g \in W^{s,q}(\Gamma) : \int_{\Gamma} \omega(y) |g(y)|^q dS_y < +\infty \right\}$$

(cf. the definition of $H_{00}^{\frac{1}{2}}(\Omega)$ in [16; Chap. 1, Thm. 11.7 (with $\mu = 0$ therein)] and the notation $H_{00}^{\frac{1}{2}}(\Gamma)$ in [3; pp. 73, 80 etc.]).

Let $\gamma : W^{1,q}(\Omega) \rightarrow W^{1-\frac{1}{q},q}(\partial\Omega)$ ($1 < q < +\infty$) denote the trace mapping (see, e. g., [8], [11], [19], [24; pp. 281-282, 329-330]). To make things clearer, we also write γ_{Ω} in place of γ .

Summarizing our preceding discussion, we have:

1° *Let $h \in W^{1,q}(\Omega)$ satisfy $\gamma(h) = 0$ a. e. on $\partial\Omega \setminus \Gamma$. Then*

$$\gamma(h)|_{\Gamma} \in W_{00}^{1-\frac{1}{q},q}(\Gamma).$$

2° *Let $g \in W_{00}^{1-\frac{1}{q},q}(\Gamma)$. Define*

$$\tilde{g} := \begin{cases} g & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega \setminus \Gamma. \end{cases}$$

Then there exists $h \in W^{1,q}(\Omega)$ such that

$$\gamma(h) = \tilde{g} \quad \text{a. e. on } \Gamma.$$

Indeed, 1° follows immediately from [1.1]. To verify 2°, we notice that our above discussion gives $\tilde{g} \in W^{1-\frac{1}{q},q}(\partial\Omega)$. The claim then follows from the inverse trace theorem (see [8], [19], [24; p. 332]). ■

[1.3] We now study the extension of **any** function $g \in W^{s,q}(\Gamma)$ by zero onto $\partial\Omega \setminus \Gamma$ (i. e. without the decay property (A1.4)).

Let $\{e_1, \dots, e_N\}$ denote the standard basis in \mathbb{R}^N . We introduce

Assumption (A) *For every $x \in \bar{\Gamma} \cap (\partial\Omega \setminus \Gamma)$ there exists*

- (i) *a Euclidean basis $\{f_1, \dots, f_N\}$ in \mathbb{R}^{N-14} ,*
- (ii) *an open cube $\Delta = \{\tau \in \mathbb{R}^{N-1} : \max\{|\tau_1|, \dots, |\tau_{N-1}|\} < \delta\}$,*

¹⁴⁾ $\{f_1, \dots, f_N\}$ originates from $\{e_1, \dots, e_N\}$ by shift and rotation.

(iii) a Lipschitz function $a : \Delta \rightarrow \mathbb{R}$

such that in terms of local coordinates $\xi \in \text{span}\{f_1, \dots, f_N\}$ ¹⁵⁾ there holds

$$1) \ x = (0, \dots, 0, a(0)),$$

$$2.1) \ \{\xi \in \mathbb{R}^d : \xi' \in \Delta, \ a(\xi') < \xi_N < a(\xi') + \delta\} \subset \Omega,$$

$$2.2) \ \{\xi \in \mathbb{R}^d : \xi' \in \Delta, \ \xi_N = a(\xi')\} \subset \partial\Omega,$$

$$2.3) \ \{\xi \in \mathbb{R}^d : \xi' \in \Delta, \ -\delta < \xi_{N-1} < 0, \ \xi_N = a(\xi')\} \subset \Gamma$$

(cf. figure 2).

For what follows we need some more notations.

$$\Delta^- := \{\xi' \in \Delta : -\delta < \xi_{N-1} < 0\},$$

$$\Delta^+ := \{\xi' \in \Delta : 0 < \xi_{N-1} < \delta\}$$

and

$$\phi(\xi) := \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{N-1} \\ a(\xi') + \xi_N \end{pmatrix}, \quad \xi = (\xi', \xi_N) \in \Delta \times (-\delta, \delta),$$

$$U := \phi(\Delta \times (-\delta, \delta)).$$

¹⁵⁾For $\xi = \text{span}\{f_1, \dots, f_N\}$ we write $\xi = (\xi', \xi_N)$, $\xi' = (\xi_1, \dots, \xi_{N-1})$.

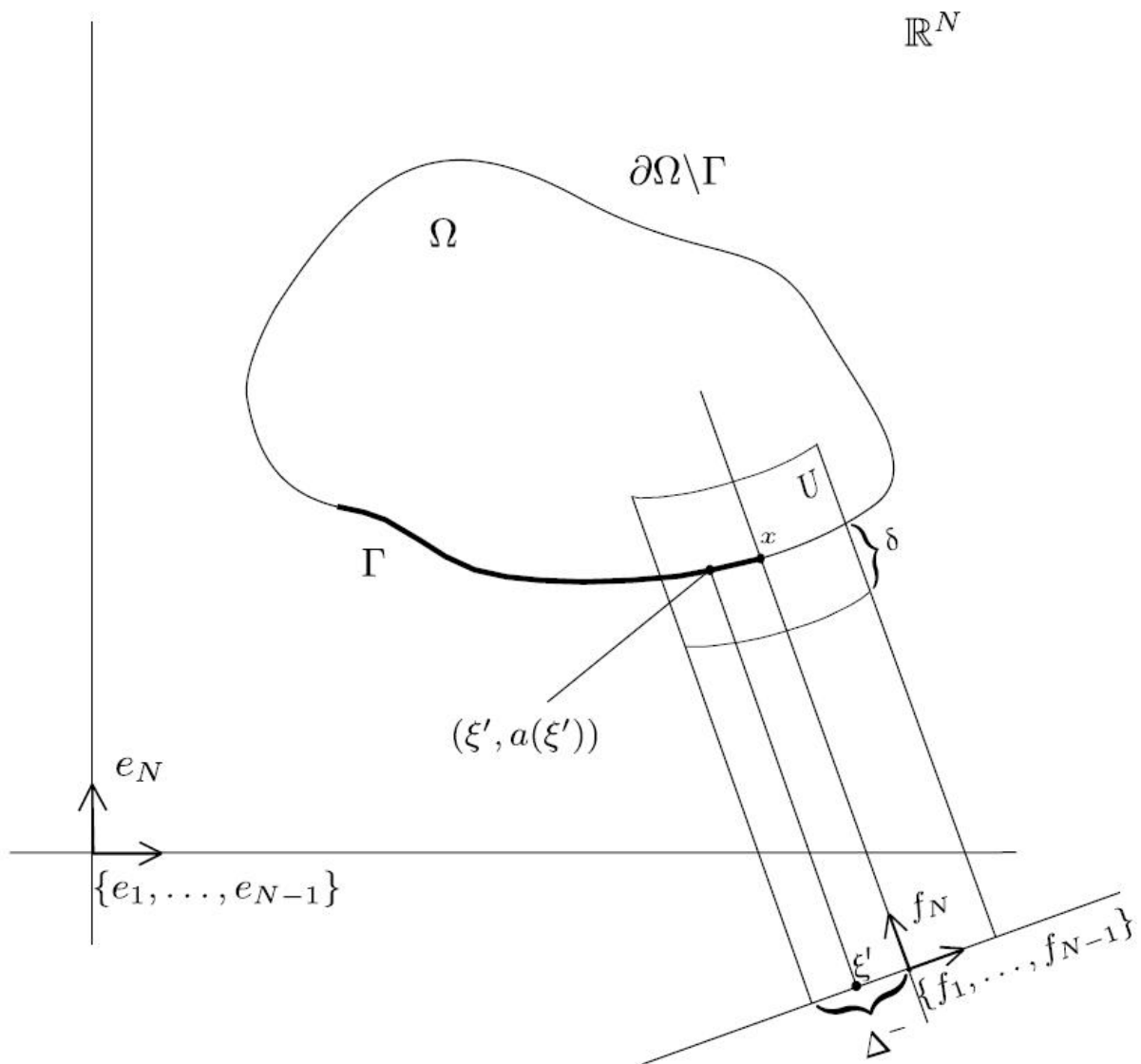


Figure 2

We obtain

$$\Delta = \Delta^- \cup \{\xi' \in \Delta : \xi_N = 0\} \cup \Delta^+,$$

$$|\xi' - \hat{\xi}'|_{\mathbb{R}^{N-1}} \leq |\phi(\xi) - \phi(\hat{\xi}')|_{\mathbb{R}^N} \leq c_0 |\xi' - \hat{\xi}'|_{\mathbb{R}^{N-1}} \quad \forall \xi, \hat{\xi} \in \Delta \times (-\delta, \delta),$$

$$\phi^{-1}(\eta) := \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{N-1} \\ \eta_N - a(\eta') \end{pmatrix}, \quad \eta = (\eta', \eta_N) \in U.$$

Then conditions 1) and 2.1)- 2.3) can be equivalently stated as follows:

- 1') $\phi(0) = (0, \dots, 0, a(0))^\top$,
- 2.1') $\phi(\Delta \times (0, \delta)) = \Omega \cap U$,
- 2.2') $\phi(\Delta \times \{0\}) = \partial\Omega \cap U$,
- 2.3') $\phi(\Delta^- \times \{0\}) = \Gamma \cap U$.

Theorem A1.1 *Let assumption (A) be satisfied and let $1 < q < +\infty$. For $g \in W^{s,q}(\Gamma)$, define*

$$\tilde{g} := \begin{cases} g & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega \setminus \Gamma. \end{cases}$$

If $s < \frac{1}{q}$, then $\tilde{g} \in W^{s,q}(\partial\Omega)$ and

$$(A1.5) \quad \|\tilde{g}\|_{W^{s,q}(\partial\Omega)} \leq c \|g\|_{W^{s,q}(\Gamma)}.$$

Proof The definition of the Lipschitz continuity of $\partial\Omega$ implies the existence of Euclidean coordinate systems $\{f_{\alpha 1}, \dots, f_{\alpha N}\}$ in \mathbb{R}^N , open cubes $\Delta_\alpha \subset \mathbb{R}^{N-1}$ and Lipschitz functions $a_\alpha : \Delta_\alpha \rightarrow \mathbb{R}$ ($\alpha = 1, \dots, m$) such that 2.1) and 2.2) hold with Δ_α and a_α in place of Δ and a , respectively (see, e. g. [8; pp. 304-306], [10; pp. 21-25], [11; pp. 5-7]). It follows $\partial\Omega \subset \bigcup_{\alpha=1}^m U_\alpha$, where

$$U_\alpha := \phi_\alpha(\Delta_\alpha \times (-\delta_\alpha, \delta_\alpha))$$

(recall $\phi_\alpha(\xi) = (\xi', a_\alpha(\xi') + \xi_N)^\top$, $\xi = (\xi', \xi_N) \in \Delta \times (-\delta, \delta)$). By 2.2), $x_\alpha = (0, \dots, 0, a_\alpha(0)) \in \partial\Omega$.

If $\Gamma \cap U_\alpha \subset \Gamma$ or $(\partial\Omega \setminus \Gamma) \cap U_\alpha \subset \partial\Omega \setminus \Gamma$ there is nothing to prove. Therefore, it suffices to consider a local representation $\{\{f_{\alpha 1}, \dots, f_{\alpha N}\}, \Delta_\alpha, a_\alpha\}$ of $\partial\Omega$ such that $x_\alpha \in \bar{\Gamma} \cap (\partial\Omega \setminus \Gamma)$. Then 2.3) of assumption (A) implies

$$\{\xi \in \mathbb{R}^d : \xi' \in \Delta, -\delta_\alpha < \xi_{N-1} < 0, \quad \xi_N = a_\alpha(\xi')\} = \Gamma \cap U_\alpha.$$

For notational simplicity, in what follows we omit the index α .

Let $g \in W^{s,q}(\Gamma)$. By 2.3),

$$\begin{aligned} \int_{\Gamma \cap U} \int_{\Gamma \cap U} \frac{|g(x) - g(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y &= \\ &= \int_{\Delta^-} \int_{\Delta^-} \frac{|g \circ \phi(\xi', a(\xi')) - g \circ \phi(\eta', a(\eta'))|^q}{|\phi(\xi', a(\xi')) - \phi(\eta', a(\eta'))|^{N-1+sq}} \sqrt{1 + |\nabla a(\xi')|^2} \sqrt{1 + |\nabla a(\eta')|^2} d\xi' d\eta' \\ &\geq c \int_{\Delta^-} \int_{\Delta^-} \frac{|g \circ \phi(\xi', a(\xi')) - g \circ \phi(\eta', a(\eta'))|^q}{|\xi' - \eta'|^{N-1+sq}} d\xi' d\eta'. \end{aligned}$$

Next, define $z(\xi') := g \circ \phi(\xi', a(\xi'))$ for a. e. $\xi' \in \Delta^-$, and

$$\tilde{z} := \begin{cases} z & \text{a. e. in } \Delta^-, \\ 0 & \text{a. e. in } \Delta^+. \end{cases}$$

Then $z \in W^{s,q}(\Delta^-)$, and

$$\tilde{z} = \tilde{g} \circ \phi \quad \text{a. e. in } \Delta, \quad \tilde{g} = \tilde{z} \circ \phi^{-1} \quad \text{a. e. in } \partial\Omega \cap U.$$

Now from [25; Thm. 3.5] (see also [16; Chap. 1, Thm. 11.4] for $q = 2$) it follows that

$$(A1.6) \quad \tilde{z} \in W^{s,q}(\Delta), \quad \|\tilde{z}\|_{W^{s,q}(\Delta)} \leq c \|z\|_{W^{s,q}(\Delta^-)}.$$

We obtain

$$\begin{aligned}
& \int_{\partial\Omega \cap U} \int_{\partial\Omega \cap U} \frac{|\tilde{g}(x) - \tilde{g}(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y = \\
& = \int_{\Delta} \int_{\Delta} \frac{|\tilde{g} \circ \phi(\xi', a(\xi')) - \tilde{g} \circ \phi(\eta', a(\eta'))|^q}{|\phi(\xi', a(\xi')) - \phi(\eta', a(\eta'))|^{N-1+sq}} \times \\
& \quad \times \sqrt{1 + |\nabla a(\xi')|^2} \sqrt{1 + |\nabla a(\eta')|^2} d\xi' d\eta' \quad [\text{by (2.1)}] \\
& \leq c_1 \int_{\Delta} \int_{\Delta} \frac{|\tilde{g} \circ \phi(\xi', a(\xi')) - \tilde{g} \circ \phi(\eta', a(\eta'))|^q}{|\xi' - \eta'|^{N-1+sq}} d\xi' d\eta' \quad [\text{by (iii)}] \\
& \leq c_2 \int_{\Delta^-} \int_{\Delta^-} \frac{|g \circ \phi(\xi', a(\xi')) - g \circ \phi(\eta', a(\eta'))|^q}{|\xi' - \eta'|^{N-1+sq}} d\xi' d\eta' \quad [\text{by (A1.6)}] \\
& \leq c_3 \int_{\Gamma \cap U} \int_{\Gamma \cap U} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y \quad [\text{by (ii) and 2.3}]
\end{aligned}$$

The proof of the theorem is now easily completed by standard arguments. ■

Remark A1.2 If $s = \frac{1}{q}$, then the statement of Theorem A1.1 fails.

2 Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ ($d = 2$ or $d = 3$) be bounded domains such that

$$\Omega_1 \cap \Omega_2 = \emptyset, \quad \Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \emptyset,$$

$\partial\Omega_i$ Lipschitz, Γ relatively open in $\partial\Omega_i$ ($i = 1, 2$)

(cf. Section 1). Let $\gamma_{\Omega_i} : W^{1,q}(\Omega_i) \rightarrow W^{1-\frac{1}{q},q}(\partial\Omega_i)$ ($1 < q < +\infty$) denote the trace mapping (cf. above). In what follows, we write $\gamma_i = \gamma_{\Omega_i}$. For $\mathbf{u}_i \in \mathbf{W}^{1,2}(\Omega_i)$ the trace $\gamma_i(\mathbf{u}_i)$ is understood componentwise. By Sobolev's embedding theorem,

$$\text{(A1.7)} \quad \begin{cases} |\mathbf{u}_i|^2 \in W^{1,q}(\Omega_i) & \text{where} \\ 1 \leq q < 2 & \text{arbitrary if } d = 2, \quad q = \frac{3}{2} \text{ if } d = 3. \end{cases}$$

Then $\gamma_i(|\mathbf{u}_i|^2) \in W^{1-\frac{1}{q},q}(\partial\Omega_i)$. ■

Let us consider

$$\mathbf{u}_i \in \mathbf{W}^{1,2}(\Omega_i), \quad \gamma_i(\mathbf{u}_i) = \mathbf{0} \quad \text{a.e. on } \partial\Omega_i \setminus \Gamma.$$

For notational simplicity, set $\mathbf{v}_i := \gamma_i(\mathbf{u}_i)$ a. e. on Γ . Then $\mathbf{v}_i \in \mathbf{W}^{\frac{1}{2},2}(\Gamma)$, $|\mathbf{v}_i|^2 \in W^{1-\frac{1}{q},q}(\Gamma)$ ¹⁶⁾ and

$$(A1.8) \quad \int_{\Gamma} |\mathbf{v}_i(y)|^{2q} \left(\int_{\partial\Omega_i \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \right) dS_y < +\infty$$

(cf. (A1.1)). To homogenize boundary condition (2.3), we have to consider the following

Problem (P) Define $g := |\mathbf{v}_1 - \mathbf{v}_2|^2$ a. e. on Γ , and

$$\tilde{g}_i := \begin{cases} g & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma. \end{cases}$$

Does there exist $\tilde{h}_i \in W^{1,q}(\Omega_i)$ such that $\gamma_i(\tilde{h}_i) = \tilde{g}_i$ a. e. on $\partial\Omega_i$?

An answer to this problem can be given by imposing the following condition on the geometry of Ω_1 and Ω_2 "near to the interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$ ":

Assumption (B) For every $y \in \Gamma$, there holds

$$\int_{\partial\Omega_1 \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x = \int_{\partial\Omega_2 \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \quad \text{for all } y \in \Gamma$$

(q as in (A1.7))

We obtain the following result.

Let assumption (B) be satisfied. Let be $\mathbf{u}_i \in \mathbf{W}^{1,2}(\Omega_i)$, $\gamma_i(\mathbf{u}_i) = \mathbf{0}$ a. e. on $\Omega_i \setminus \Gamma$ ($i = 1, 2$). Set $\mathbf{v}_i := \gamma_i(\mathbf{u}_i)$ a. e. on Γ . If

$$(A1.9) \quad |\mathbf{v}_1 - \mathbf{v}_2|^2 \in W^{1-\frac{1}{q},q}(\Gamma) \quad (q \text{ as in (A1.7)}),$$

then there exists $\tilde{h}_i \in W^{1,q}(\Omega_i)$ such that

$$\gamma_i(\tilde{h}_i) = \tilde{g}_i \quad \text{a.e. on } \partial\Omega_i.$$

¹⁶⁾The definition of the trace mapping implies

$$(\gamma(\varphi))^2 = (\varphi|_{\Gamma})^2 = \varphi^2|_{\Gamma} = \gamma(\varphi^2)$$

for every $\varphi \in C^1(\bar{\Omega})$. Thus, by approximation

$$|\mathbf{v}_i|^2 = \sum_{l=1}^d (\gamma_i(u_{il}))^2 = \sum_{l=1}^d \gamma_i(u_{il}^2) = \gamma_i\left(\sum_{l=1}^d u_{il}^2\right) = \gamma_i(|\mathbf{u}_i|^2).$$

Indeed, combining (A1.8) and assumption (B) we find

$$\int_{\Gamma} |(\mathbf{v}_1 - \mathbf{v}_2)(y)|^{2q} \left(\int_{\partial\Omega_i \setminus \Gamma} \frac{1}{|x - y|^{d-2+q}} dS_x \right) dS_y < +\infty \quad (i = 1, 2).$$

Observing (A1.9) we see that (A1.3) and (A1.4) are satisfied with $g = |\mathbf{v}_1 - \mathbf{v}_2|^2$, $N = d$, $s = 1 - \frac{1}{q}$ and $\Omega = \Omega_i$. The claim follows from 1.2 above. \blacksquare

It is easily verified that this result continues to hold for $G_i(|\mathbf{v}_1 - \mathbf{v}_2|^2)$ in place of $|\mathbf{v}_1 - \mathbf{v}_2|^2$.

We notice that assumption (B) is satisfied if Ω_1 and Ω_2 obey an appropriate symmetry property with respect to Γ .

Remark A1.2 Assumption (A1.9) is equivalent to

$$(A1.9') \quad \mathbf{v}_1 \cdot \mathbf{v}_2 \in W^{1-\frac{1}{q},q}(\Gamma).$$

This is readily seen when observing the elementary identity

$$|\mathbf{a} - \mathbf{b}|^2 - |\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 = |\mathbf{a}|^2 - |\hat{\mathbf{a}}|^2 + (|\mathbf{b}|^2 - |\hat{\mathbf{b}}|^2) - 2(\mathbf{a} \cdot \mathbf{b} - \hat{\mathbf{a}} \cdot \hat{\mathbf{b}})$$

$$(\mathbf{a}, \hat{\mathbf{a}}, \mathbf{b}, \hat{\mathbf{b}} \in \mathbb{R}^d).$$

Remark A1.3 We notice that (A1.9') is true in case $d = 2$. To see this, first observe that $W^{\frac{1}{2},2}(\partial\Omega_i) \subset L^r(\partial\Omega_i)$ ($1 \leq r < +\infty$ arbitrary). We obtain, for every $1 \leq q < 2$,

$$\begin{aligned} & \int_{\Gamma} \int_{\Gamma} \frac{|\mathbf{v}_i(x) - \mathbf{v}_i(y)|^q}{|x - y|^q} |\mathbf{v}_j(x)|^q dS_x dS_y \leq \\ & \leq \int_{\Gamma} \left(\int_{\Gamma} \frac{|\mathbf{v}_i(x) - \mathbf{v}_i(y)|^2}{|x - y|^2} dS_x \right)^{\frac{q}{2}} \left(\int_{\Gamma} |\mathbf{v}_j(x)|^{\frac{2q}{2-q}} dS_x \right)^{\frac{2-q}{2}} dS_y \\ & \leq (\text{mes } \Gamma)^{\frac{2-q}{2}} \|\mathbf{v}_i\|_{\mathbf{W}^{\frac{1}{2},2}(\Gamma)}^q \|\mathbf{v}_j\|_{\mathbf{L}^{\frac{2q}{2-q}}(\Gamma)}^q \end{aligned}$$

($i, j = 1, 2; i \neq j$). Whence (A1.9').

We obtain: if $d = 2$ and assumption (B) holds, then problem (\mathcal{P}) has a solution. \blacksquare

Theorem A1.2 Suppose that $\Gamma \cap (\partial\Omega_i \setminus \Gamma)$ ($i = 1, 2$) satisfies assumption (A). Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{W}^{\frac{1}{2},2}(\Gamma)$. Define $g := |\mathbf{v}_1 - \mathbf{v}_2|^2$ a. e. on Γ , and

$$\tilde{g}_i := \begin{cases} g & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma. \end{cases}$$

Then

$$\tilde{g}_i \in \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1-\frac{1}{q},q}(\partial\Omega_i),$$

$$\|\tilde{g}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \leq c \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^{\frac{1}{2},2}(\Gamma)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^r(\Gamma)},$$

where

$$\begin{aligned} c &= c(q) \rightarrow +\infty \quad \text{as } q \rightarrow \frac{d}{d-1}, \quad \left(1 \leq q < \frac{d}{d-1}\right), \\ r &= \frac{2q}{2-q} \quad \text{if } d=2, \quad r=4 \quad \text{if } d=3. \end{aligned}$$

Proof $\boxed{d=2}$ First, notice $W^{\frac{1}{2},2}(\partial\Omega) \subset L^r(\partial\Omega)$ ($1 \leq r < +\infty$) continuously.

Observing that

$$\left| |\mathbf{a} - \mathbf{b}|^2 - |\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 \right| \leq |\mathbf{a} - \mathbf{b} - (\hat{\mathbf{a}} - \hat{\mathbf{b}})| |\mathbf{a} - \mathbf{b} + (\hat{\mathbf{a}} - \hat{\mathbf{b}})|, \quad \mathbf{a}, \hat{\mathbf{a}}, \mathbf{b}, \hat{\mathbf{b}} \in \mathbb{R}^N,$$

we obtain by the aid of Hölder's inequality, for every $1 \leq q < 2$,

$$\begin{aligned} & \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x - y|^q} dS_x dS_y \leq \\ & \leq \left(\int_{\Gamma} \int_{\Gamma} \frac{|\mathbf{v}_1(x) - \mathbf{v}_2(x) - (\mathbf{v}_1(y) - \mathbf{v}_2(y))|^2}{|x - y|^2} dS_x dS_y \right)^{\frac{q}{2}} \times \\ & \quad \times \left(\int_{\Gamma} \int_{\Gamma} |\mathbf{v}_1(x) - \mathbf{v}_2(x) + (\mathbf{v}_1(y) - \mathbf{v}_2(y))|^{\frac{2q}{2-q}} dS_x dS_y \right)^{\frac{2-q}{q}} \\ & \leq c \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^{\frac{1}{2},2}(\Gamma)}^q \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^{\frac{2q}{2-q}}(\Gamma)}^q. \end{aligned}$$

Thus, $g \in W^{1-\frac{1}{q},q}(\Gamma)$ and

$$\begin{aligned}
\|g\|_{W^{1-\frac{1}{q}}(\Gamma)}^q &\leq \left(\int_{\Gamma} |\mathbf{v}_1 - \mathbf{v}_2|^{\frac{2q}{2-q}} \right)^{2-q} + \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x - y|^q} dS_x dS_y \\
&\leq c \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^{\frac{1}{2}}(\Gamma)}^q \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{L}^{\frac{2q}{2-q}}(\Gamma)}^q.
\end{aligned}$$

On the other hand, Theorem A1.1 (with $\Omega = \Omega_i$, $s = 1 - \frac{1}{q}$, $s < \frac{1}{q}$) gives

$$\tilde{g}_i \in W^{1-\frac{1}{q}}(\partial\Omega_i), \quad \|\tilde{g}_i\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \leq c \|g\|_{W^{1-\frac{1}{q},q}(\Gamma)}.$$

Whence the claim.

$d = 3$ Then $W^{\frac{1}{2},2}(\partial\Omega) \subset L^4(\partial\Omega)$ continuously. Hence $g \in L^2(\Gamma)$. We divide the proof into two steps.

Step 1 For every $0 < \delta < 1$, there holds

$$(A1.10) \quad \begin{cases} g \in W^{\frac{1-\delta}{2}, \frac{4}{3}}(\Gamma), \\ \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^{\frac{4}{3}}}{|x - y|^{2+\frac{2(1-\delta)}{3}}} dS_x dS_y \leq c \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^{\frac{1}{2},2}(\Gamma)}^{\frac{4}{3}} \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{L}^4(\Gamma)}^{\frac{4}{3}}. \end{cases}$$

Indeed, with the help of the above inequality for $\mathbf{a}, \hat{\mathbf{a}}, \mathbf{b}, \hat{\mathbf{b}} \in \mathbb{R}^N$ and Hölder's inequality we find

$$\begin{aligned}
&\int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^{\frac{4}{3}}}{|x - y|^{2+\frac{2(1-\delta)}{3}}} dS_x dS_y \leq \\
&\leq c \int_{\Gamma} \int_{\Gamma} \frac{\left| |\mathbf{v}_1(x) - \mathbf{v}_2(x)|^2 - |\mathbf{v}_1(y) - \mathbf{v}_2(y)|^2 \right|^{\frac{4}{3}}}{|x - y|^{2+\frac{2(1-\delta)}{3}}} dS_x dS_y \\
&\leq c \int_{\Gamma} \int_{\Gamma} \frac{|\mathbf{v}_1(x) - \mathbf{v}_2(x) - (\mathbf{v}_1(y) - \mathbf{v}_2(y))|^{\frac{4}{3}}}{|x - y|^2} \times \\
&\quad \times \frac{|\mathbf{v}_1(x) - \mathbf{v}_2(x) + (\mathbf{v}_1(y) - \mathbf{v}_2(y))|^{\frac{4}{3}}}{|x - y|^{\frac{2(1-\delta)}{3}}} dS_x dS_y
\end{aligned}$$

$$(A1.11) \quad \leq c \|\mathbf{v}_1 - \mathbf{v}_2\|_{W^{\frac{1}{2},2}(\Gamma)}^{\frac{4}{3}} \times \\ \times \left(\int_{\Gamma} \int_{\Gamma} \frac{(|\mathbf{v}_1(x) - \mathbf{v}_2(x)| + |\mathbf{v}_1(y) - \mathbf{v}_2(y)|)^4}{|x - y|^{2(1-\delta)}} dS_x dS_y \right)^{\frac{1}{3}}$$

(notice that $W^{\frac{1}{2},2}(\Gamma) \subset L^4(\Gamma)$).

Next, by elementary integral calculus it is easily seen that there exists a positive constant K_0 such that

$$\int_{\partial\Omega_i} \frac{1}{|x - y|^{2(1-\delta)}} dS_y \leq K_0 \quad \forall x \in \partial\Omega_i \quad (i = 1, 2)$$

($K_0 = K_0(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0$). Then the second double integral on the right hand side of (A1.11) can be estimated as follows

$$\begin{aligned} & \int_{\Gamma} \int_{\Gamma} \frac{(|\mathbf{v}_1(x) - \mathbf{v}_2(x)| + |\mathbf{v}_1(y) - \mathbf{v}_2(y)|)^4}{|x - y|^{2(1-\delta)}} dS_x dS_y \leq \\ & \leq 16 \int_{\Gamma} |\mathbf{v}_1(x) - \mathbf{v}_2(x)|^4 \left(\int_{\Gamma} \frac{1}{|x - y|^{2(1-\delta)}} dS_y \right) dS_x \\ & \quad + 16 \int_{\Gamma} |\mathbf{v}_1(y) - \mathbf{v}_2(y)|^4 \left(\int_{\Gamma} \frac{1}{|x - y|^{2(1-\delta)}} dS_x \right) dS_y \\ & \leq 32K_0 \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^4(\Gamma)}^4. \end{aligned}$$

Inserting this estimate into (A1.11) we find (A1.10) ($c = c(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0$).

Step 2 From Theorem A1.1 (with $\Omega = \Omega_i$, $s = \frac{1-\delta}{2}$, $q = \frac{4}{3}$) and (A1.10) it follows that

$$\begin{aligned} \tilde{g} & \in W^{\frac{1-\delta}{2}, \frac{4}{3}}(\partial\Omega_i), \\ \|\tilde{g}\|_{W^{\frac{1-\delta}{2}, \frac{4}{3}}(\partial\Omega_i)} & \leq c \|g\|_{W^{\frac{1-\delta}{2}, \frac{4}{3}}(\Gamma)} = \\ & = c \left(\|g\|_{L^{\frac{4}{3}}(\Gamma)}^{\frac{4}{3}} + \int_{\Gamma} \int_{\Gamma} \frac{|\tilde{g}(x) - \tilde{g}(y)|^{\frac{4}{3}}}{|x - y|^{2 + \frac{2(1-\delta)}{3}}} dS_x dS_y \right)^{\frac{3}{4}} \\ & \leq (\|\mathbf{v}_1 - \mathbf{v}_2\|_{L^{\frac{4}{3}}(\Gamma)}^2 + \|\mathbf{v}_1 - \mathbf{v}_2\|_{W^{\frac{1}{2},2}(\Gamma)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^4(\Gamma)}) \end{aligned}$$

$$(A1.12) \quad \leq (\|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^{\frac{1}{2},2}(\Gamma)}^2 \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^4(\Gamma)}).$$

To proceed, we notice the continuous embedding

$$(A1.13) \quad W^{\frac{1-\delta}{2}, \frac{4}{3}}(\partial\Omega_i) \subset W^{\frac{1-\delta}{2}-\alpha, \frac{4}{3-2\alpha}}(\partial\Omega_i) \quad \left(0 < \alpha < \frac{1-\delta}{2}\right)$$

(see, e. g., [1], [25; p. 328, $n = d - 1 = 2$ in (8)]).

Now, consider q such that $\frac{4}{3} < q < \frac{3}{2}$. Define

$$\delta := \frac{2(3-2q)}{q}, \quad \alpha := \frac{1-2\delta}{6}.$$

It follows

$$\frac{1-\delta}{2} - \alpha = 1 - \frac{1}{q}, \quad \frac{4}{3-2\alpha} = q.$$

By combining (A1.12) and (A1.13) we obtain the statement of Theorem A1.2 when $d = 3$. ■

Corollary A1.1 *Suppose that $\Gamma \cap (\partial\Omega_i \setminus \Gamma)$ ($i = 1, 2$) satisfies assumption (A). Let be $\mathbf{u}_i \in \mathbf{W}^{1,2}(\Omega_i)$ such that*

$$\gamma_i(\mathbf{u}_i) = \mathbf{0} \quad \text{a. e. on } \partial\Omega_i \setminus \Gamma.$$

Define

$$\tilde{h}_i := \begin{cases} G_i(|\gamma_1(\mathbf{u}_1) - \gamma_2(\mathbf{u}_2)|^2) & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma \end{cases}$$

(G_i as in (1.6); $i = 1, 2$).

Then, for every $1 \leq q < \frac{d}{d-1}$,

$$\tilde{h}_i \in W^{1-\frac{1}{q}, q}(\partial\Omega_i), \quad \|\tilde{h}_i\|_{W^{1-\frac{1}{q}, q}(\partial\Omega_i)} \leq c \sum_{j=1}^2 \|\mathbf{u}_j\|_{\mathbf{W}^{1,2}(\Omega_j)}^2,$$

where $c = c(q) \rightarrow +\infty$ as $q \rightarrow \frac{d}{d-1}$.

Proof As above, for notational simplicity, set $\mathbf{v}_i := \gamma_i(\mathbf{u}_i)$ and $h_i := G_i(|\mathbf{v}_1 - \mathbf{v}_2|^2)$ a. e. on Γ ($i = 1, 2$). Then

$$\tilde{h}_i := \begin{cases} h_i & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma \end{cases}$$

and

$$|h_i(x) - h_i(y)| \leq c_0 \left| |\mathbf{v}_1(x) - \mathbf{v}_2(x)|^2 - |\mathbf{v}_1(y) - \mathbf{v}_2(y)|^2 \right| \quad \text{for a. e. } x, y \in \Gamma.$$

It is readily seen that the proof of Theorem A1.2 can be repeated word by word with h_i and \tilde{h}_i in place of g and \tilde{g}_i , respectively. We obtain

$$\tilde{h}_i \in \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1-\frac{1}{q}, q}(\partial\Omega_i),$$

$$\|\tilde{h}_i\|_{W^{1-\frac{1}{q}, q}(\partial\Omega_i)} \leq c \|\mathbf{v}_1 - \mathbf{v}_2\|_{\mathbf{W}^{\frac{1}{2}}(\Gamma)} \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^r(\Gamma)},$$

where r is as in Theorem A1.2.

Combining this and the continuity of the trace mapping $\gamma_i : W^{1,2}(\Omega_i) \rightarrow W^{\frac{1}{2},2}(\partial\Omega_i)$ we get the assertion of the corollary.

Appendix 2. The inhomogeneous Dirichlet problem for the Poisson equation with right hand side in L^1

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with boundary $\partial\Omega \in \mathcal{C}^1$. We consider the following boundary value problem:

$$(A2.1) \quad -\Delta u = f \quad \text{in } \Omega,$$

$$(A2.2) \quad u = g \quad \text{on } \partial\Omega.$$

Our basic existence result concerning weak solutions to this problem is

Theorem A2.1 *Assume*

$$f \in L^1(\Omega), \quad g \in W^{1-\frac{1}{q}, q}(\partial\Omega) \quad \left(1 < q < \frac{N}{N-1}\right)$$

Then, there exists exactly one $u \in W^{1,q}(\Omega)$ such that

$$(A2.3) \quad \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \quad \forall \varphi \in W_0^{1,q'}(\Omega),$$

$$(A2.4) \quad u = g \quad \text{on} \quad \partial\Omega,$$

$$(A2.5) \quad \|u\|_{W^{1,q}} \leq c(\|f\|_{L^1} + \|g\|_{W^{1-\frac{1}{q},q}})$$

Moreover, for every $\Omega' \subset\subset \Omega$ and every $\delta > 0$ there holds

$$(A2.6) \quad \left\{ \begin{array}{l} \frac{\nabla u}{(1+|u|)^{\frac{1+\delta}{2}}} \in \mathbf{L}^2(\Omega'), \\ \int_{\Omega'} \frac{|\nabla u|^2}{(1+|u|)^{1+\delta}} \leq \frac{c}{\delta} (\|f\|_{L^1} + \|g\|_{W^{1-\frac{1}{q},q}}) \\ \text{where } c \rightarrow +\infty \text{ as } \text{dist}(\Omega', \partial\Omega) \rightarrow 0. \end{array} \right.$$

If, in addition, $f \in L_{\text{loc}}^r(\Omega)$ ($r > 1$) then

$$(A2.7) \quad u \in W_{\text{loc}}^{2,r}(\Omega).$$

Proof We begin by noting the following result. For every $1 < q < +\infty$ there exists a positive constant C_q such that, for any $v \in W_0^{1,q}(\Omega)$,

$$(A2.8) \quad \|\nabla v\|_{L^q} \leq C_q \sup \left\{ \frac{\int_{\Omega} \nabla v \cdot \nabla \varphi}{\|\nabla \varphi\|_{L^{q'}}}; \quad \varphi \in W_0^{1,q'}(\Omega), \quad \varphi \neq 0 \right\}$$

(see. [21; Thm. 4.2, p. 191]).

Next, by the inverse trace theorem, there exists $h \in W^{1,q}(\Omega)$ such that

$$\gamma(h) = g \quad \text{a. e. on} \quad \partial\Omega, \quad \|h\|_{W^{1,q}} \leq c\|g\|_{W^{1-\frac{1}{q},q}}.$$

Then we can find functions $f_m, h_m \in C^\infty(\bar{\Omega})$ ($m \in \mathbb{N}$) such that

$$f_m \rightarrow f \quad \text{strongly in} \quad L^1(\Omega), \quad h_m \rightarrow h \quad \text{strongly in} \quad W^{1,q}(\Omega)$$

as $m \rightarrow \infty$. The Riesz representation theorem for linear continuous functionals on the Hilbert space $W_0^{1,2}(\Omega)$ provides the existence and uniqueness of a $v_m \in W_0^{1,2}(\Omega)$ satisfying

$$(A2.9) \quad \int_{\Omega} \nabla v_m \cdot \nabla \varphi = \int_{\Omega} (f_m \varphi + (\partial_i h_m) \partial_i \varphi) \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Now, let $1 < q < \frac{N}{N-1}$. Observing that $W^{1,q'}(\Omega) \subset C(\bar{\Omega})$ we obtain

$$\left| \int_{\Omega} (f_m \varphi + (\partial_i h_m) \partial_i \varphi) \right| \leq c(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}}) \|\varphi\|_{W^{1,q'}} \quad \forall \varphi \in W_0^{1,q'}(\Omega).$$

Combining this estimate and (A2.8), (A2.9) gives

$$\|\nabla v_m\|_{L^q} \leq c(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}}).$$

Define $u_m := v_m + h_m$ ($m \in \mathbb{N}$). Then $u_m \in W^{1,2}(\Omega)$ and

$$(A2.10) \quad \int_{\Omega} \nabla u_m \cdot \nabla \varphi = \int_{\Omega} f_m \varphi \quad \forall \varphi \in W_0^{1,2}(\Omega),$$

$$(A2.11) \quad u_m = h_m \quad \text{a. e. on } \partial\Omega \quad [\text{in the sense of traces}],$$

$$(A2.12) \quad \|\nabla u_m\|_{L^q} \leq c(\|f_m\|_{L^q} + \|h_m\|_{W^{1,q}}).$$

From (A2.12) we conclude (by passing to a subsequence if necessary) that $u_m \rightarrow u$ weakly in $W^{1,q}(\Omega)$ as $m \rightarrow \infty$. By a routine argument, $u = g$ a. e. on $\partial\Omega$ (in the sense of traces). The passage to the limit $m \rightarrow \infty$ in (A2.10), (A2.11) gives (A2.3), (A2.4), respectively. Finally, taking the $\liminf_{m \rightarrow \infty}$ on both sides of (A2.12) provides (A2.5).

The uniqueness of u follows from (A2.5).

To prove the interior estimate (A2.6), let $\delta > 0$. We consider the function

$$\phi(t) = \phi_{\delta}(t) := \left(1 - \frac{1}{(1+|t|)^{\delta}}\right) \text{sign } t, \quad t \in \mathbb{R}.$$

Clearly,

$$|\phi(t)| \leq 1, \quad \phi'(t) = \frac{\delta}{(1+|t|)^{1+\delta}} \quad \forall t \in \mathbb{R}.$$

Let $\zeta \in C_c^1(\Omega)$ be a cut-off function for Ω' , i. e. $\zeta \equiv 1$ on Ω' and $0 \leq \zeta \leq 1$ in Ω . Then the function $\varphi = \phi(u_m)\zeta^2$ is admissible in (A2.10). By (A2.12),

$$\begin{aligned} \delta \int_{\Omega'} \frac{|\nabla u_m|^2}{(1+|u_m|)^{1+\delta}} &\leq \|f_m\|_{L^1} + 2 \max_{\Omega} |\nabla \zeta| \int_{\Omega} |\nabla u_m| \\ &\leq \|f_m\|_{L^1} + 2 \max_{\Omega} |\nabla \zeta| (\text{mes } \Omega)^{\frac{1}{q'}} \cdot c \left(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}} \right). \end{aligned}$$

Thus,

$$(A2.13) \quad \int_{\Omega'} \frac{|\nabla u_m|^2}{(1 + |u_m|)^{1+\delta}} \leq \frac{C}{\delta} \quad \forall m \in \mathbb{N} \quad (C = \text{const}).$$

As above, we may assume that $u_m \rightarrow u$ weakly in $W^{1,q}(\Omega)$ and, in addition, $u_m \rightarrow u$ a. e. in Ω . These convergence properties together with (A2.13) imply

$$\frac{\nabla u_m}{(1 + |u_m|)^{\frac{1+\delta}{2}}} \rightarrow \frac{\nabla u}{(1 + |u|)^{\frac{1+\delta}{2}}} \quad \text{weakly in } \mathbf{L}^2(\Omega') \quad \text{as } m \rightarrow \infty.$$

Whence (2.6).

To prove (A2.7), we first note that $W^{1,q}(\Omega) \subset L^{\frac{Nq}{N-q}}(\Omega)$. Now, let B_R be a ball such that $B_{2R} \subset \Omega$. Let $1 < r \leq \frac{Nq}{N-q}$. Then $u \in L^r(B_{2R})$ and

$$\left| \int_{B_{2R}} u \Delta \varphi \right| \leq \|f\|_{L^r(B_{2R})} \|\varphi\|_{L^r(B_{2R})} \quad \forall \varphi \in C_c^\infty(B_{2R}) \quad [\text{by (A2.3)}].$$

From [20; Thm. 9.5 (3), p. 144] it follows

$$u \in W^{2,r}(B_R), \quad \|u\|_{W^{2,r}(B_R)} \leq c(\|f\|_{L^r(B_{2R})} + \|u\|_{L^r(B_{2R})}).$$

Hence, (A2.7) holds for all values of r satisfying $1 < r \leq \frac{Nq}{N-q}$. By a bootstrapping argument, (A2.7) can be proved for any $r > \frac{Nq}{N-q}$. \blacksquare

Remark A2.1 We notice that the existence and uniqueness result stated in Theorem A2.1, follows from the L^p -theory of linear elliptic boundary value problems developed in [15], provided the boundary $\partial\Omega$ is sufficiently smooth. Theorem A2.1 is also an immediate consequence of [20; Thm. 10.7, pp. 181-182; $\partial\Omega \in \mathcal{C}^1$].

On the other hand, the existence of a weak solution $u \in \bigcap_{1 < q < \frac{N}{N-1}} W_0^{1,q}(\Omega)$ to linear elliptic equations in divergence form with bounded measurable coefficients, right hand sides in L^1 and zero boundary condition has been proved in [23] by a duality argument.

Remark A2.2 Our approximation procedure for solving boundary value problem (A2.1), (A2.2) permits to prove additional properties of the weak solution $u \in W^{1,q}(\Omega)$ ($1 < q < \frac{N}{N-1}$) (for instance, the interior estimate (A2.6)). Moreover, we have

Theorem A2.2 *Let the assumptions of Theorem A2.1 hold. Let $u \in W^{1,q}(\Omega)$ satisfy (A2.3)-(A2.5). Then*

1° *if $f \geq 0$ a. e. in Ω and $g \geq 0$ a. e. on $\partial\Omega$, then*

$$u \geq 0 \quad \text{a. e. in } \Omega;$$

2° if $f \in L^r_{\text{loc}}(\Omega)$ ($r > \frac{N}{2}$), then

$$\text{ess sup}_{\Omega'} |u| < +\infty \quad \forall \Omega' \subset\subset \Omega;$$

3° if $f \in L^r(\Omega)$ ($r > \frac{N}{2}$), $\text{ess sup}_{\partial\Omega} |g| < +\infty$, then

$$\text{ess sup}_{\Omega} |u| < +\infty.$$

This theorem can be proved by the methods developed in [5] and [23].

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